The zeta function of H^2 counting all subrings

1 Presentation

H^2 has presentation

$$\langle x_1, x_2, y_1, y_2, z_1, z_2 \mid [x_1, y_1] = z_1, [x_2, y_2] = z_2 \rangle .$$

H^2 has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Gareth Taylor. It is

$$\begin{aligned}\zeta_{H^2,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-4)^2\zeta_p(2s-5)^2\zeta_p(3s-5) \\ & \times \zeta_p(3s-7)\zeta_p(3s-8)W(p, p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}& 1 - X^4Y^3 - 3X^5Y^3 - X^7Y^3 + X^5Y^4 - X^9Y^4 - X^8Y^5 + 3X^9Y^5 - 2X^{11}Y^5 \\ & + X^{10}Y^6 + 3X^{11}Y^6 + 3X^{12}Y^6 + 2X^{13}Y^6 + X^{14}Y^6 - X^{14}Y^7 + X^{15}Y^7 \\ & - X^{14}Y^8 + X^{15}Y^8 - X^{15}Y^9 - 2X^{16}Y^9 - 3X^{17}Y^9 - 3X^{18}Y^9 - X^{19}Y^9 \\ & + 2X^{18}Y^{10} - 3X^{20}Y^{10} + X^{21}Y^{10} + X^{20}Y^{11} - X^{24}Y^{11} + X^{22}Y^{12} \\ & + 3X^{24}Y^{12} + X^{25}Y^{12} - X^{29}Y^{15}.\end{aligned}$$

$\zeta_{H^2}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\left.\zeta_{H^2,p}(s)\right|_{p\to p^{-1}} = p^{15-6s}\zeta_{H^2,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{H^2}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-4)^2\zeta_p(2s-5)^2\zeta_p(3s-5)\zeta_p(3s-7) \times \zeta_p(3s-8)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$W_1(X,Y) = 1 - X^7Y^3 + X^{14}Y^6,$$
$$W_2(X,Y) = 1 - X^{10}Y^5,$$
$$W_3(X,Y) = -1 - X^5Y^4.$$

The ghost is friendly.

6 Natural boundary

$\zeta_{H^2}(s)$ has a natural boundary at $\Re(s) = 7/3$, and is of type III.